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# The Lee–Friedrichs Model: Continuous Limit and Decoherence

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**Abstract** We analyze the thermodynamic limit of the Hamiltonian, states and observables, of a system containing an oscillator interacting with a thermal bath We use the results to a compare environment and self induced decoherence.

## 1 Introduction

In previous papers we developed a formalism suitable to describe the spontaneous decay process of unstable quantum systems with continuous spectrum [1, 2]. In this approach the class of relevant observables have a diagonal singularity in the energy representation, and the states are defined as functionals acting on the space of observables.

This formalism was later proved to be useful to describe the interaction of an atom and the electromagnetic radiation in a thermal state. In this case the diagonal singularities were moved from the observables to the state functionals, while the relevant local observables were represented by regular functions [3].

In both cases we have been able to find a well defined "final" state functional of the system ($t \to \infty$), for the irreversible processes.

This formalism was also used to describe the decoherence process [4], following an approach that we have called self induced decoherence (SID) [5–12].

Later on, Schlosshauer [13] obtained numerical solutions of the spin bath model developed by Zurek [14], for increasing values of the number of spins in the bath, searching for a time evolution producing the vanishing of the off diagonal elements of the total density operator in the energy representation. The result of this search was negative: it was not

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found the vanishing of the off-diagonal elements due to time evolution. On the other hand self induced decoherence, i.e. the vanishing of the off diagonal elements, is found in our approach for close systems with continuous spectrum. This was interpreted by Schlosshauer as a contradiction with the numerical results, and therefore as a strong limitation for our approach.

Nevertheless we have shown in [12] that even if the result is correct, it cannot be used against SID since the bath of the model of papers [13] and [14] has not a self interaction, in which case the whole system does not decohere in a finite time (see also [15]).

On the other hand, another feature of paper [13] also deserves a criticism: the spin bath model does not reduce to a well defined continuous model for increasing number of spins in the bath. Therefore the Riemann–Lebesgue theorem, the essential tool of SID, is not valid.

This last fact motivated us to try to get to a better understanding of the continuous limit of a discrete model. In this paper we reanalyze the well known Lee–Friedrichs toy model that mimics the atom-radiation interaction process, for the case of the radiation in a thermal state. The discrete spectrum case was analyzed by Gaioli et al. [16], and the continuous case in the thermodynamic limit was previously developed in our paper [3]. In this case we gave for the Lee–Friedrichs model a detailed description of the construction of the continuous limit.

In Sect. 2 the model is presented for the discrete case, and the continuous limit of the Hamiltonian is obtained, which is diagonalizable through an analytic expression of a Bogolyubov transformation. In Sect. 3 we define the suitable class of observables. In Sect. 4 we obtain the general form for the state functionals, and in Sect. 5 we obtain the time evolution and the decoherence. The conclusions are given in Sect. 6.

## 2 The Model

Let us consider a thermal bath of oscillators, which is the scalar version of the electromagnetic radiation in a cubic box. The Hamiltonian is given by

$$\hat{H}_{\text{bath}} = \sum_{\bar{p}} \omega_{\bar{p}} \hat{a}^{\dagger}_{\bar{p}} \hat{a}_{\bar{p}}, \qquad [\hat{a}_{\bar{p}}, \hat{a}^{\dagger}_{\bar{p}'}] = \delta_{\bar{p}\,\bar{p}'}, \tag{1}$$

where $\omega_{\bar{p}} = |\bar{p}|$, $\bar{p} = \frac{2\pi}{L}(n_x, n_y, n_z)$, $L$ is the size of the cube, and $n_x$, $n_y$, $n_z$ are integer numbers ($c = \hbar = 1$).

The thermal bath interacts with an oscillator, having the Hamiltonian

$$\hat{H}_{\text{osc}} = \Omega \hat{b}^{\dagger} \hat{b}, \qquad [\hat{b}, \hat{b}^{\dagger}] = 1. \tag{2}$$

This system can be considered as a toy model for an atom with energy levels $n\Omega$ ($n = 0, 1, 2, \ldots$).

The space to describe the interaction is the tensor product of the Hilbert spaces of both the oscillator and the bath. Therefore, it is convenient to replace the creation and annihilation operators in (1) and (2), by

$$\begin{aligned} &\hat{b}^{\dagger} \to \hat{\hat{b}}^{\dagger} \equiv \hat{b}^{\dagger} \otimes \hat{I}_{\text{bath}}, \qquad \hat{a}^{\dagger}_{\bar{p}} \to \hat{\hat{a}}^{\dagger}_{\bar{p}} \equiv \hat{I}_{\text{osc}} \otimes \hat{a}^{\dagger}_{\bar{p}}, \\ &\hat{b} \to \hat{\hat{b}} \equiv \hat{b} \otimes \hat{I}_{\text{bath}}, \qquad \hat{a}_{\bar{p}} \to \hat{\hat{a}}_{\bar{p}} \equiv \hat{I}_{\text{osc}} \otimes \hat{a}_{\bar{p}}, \end{aligned} \tag{3}$$

where $\hat{I}_{\text{osc}}$ and $\hat{I}_{\text{bath}}$ are the identity operators in the Hilbert spaces of the oscillator and the bath. The double tildes have been used to denote operators acting on the tensor product Hilbert space.

The interaction Hamiltonian is

$$H_{\text{int}} = \sum_{\bar{p}} g_{\bar{p}} (\hat{\hat{b}}^{\dagger} \hat{\hat{a}}_{\bar{p}} + \hat{\hat{a}}^{\dagger}_{\bar{p}} \hat{\hat{b}}). \tag{4}$$

When the size $L$ of the cubic box is very big, the vector $\bar{p}$ becomes a continuous variable. Using the following well known transformations

$$\left(\frac{2\pi}{L}\right)^3 \sum_{\bar{p}} \to \int d^3 p, \qquad \left(\frac{2\pi}{L}\right)^3 \delta_{\bar{p}\bar{p}'} \to \delta^3(\bar{p}-\bar{p}'), \qquad \left(\frac{L}{2\pi}\right)^{3/2} \hat{a}^{\dagger}_{\bar{p}} \to \hat{a}^{\dagger}(\bar{p}), \tag{5}$$

we can obtain the continuous version of the expressions (1):

$$\hat{H}_{\text{bath}} = \int d^3 p \omega_{\bar{p}} \hat{\hat{a}}^{\dagger}(\bar{p}) \hat{\hat{a}}(\bar{p}), \qquad [\hat{\hat{a}}(\bar{p}), \hat{\hat{a}}^{\dagger}(\bar{p}')] = \delta^3(\bar{p}-\bar{p}'). \tag{6}$$

For a well defined limit of the interaction Hamiltonian of (4), the coupling coefficients $g_{\bar{p}}$ should have the asymptotic form $g_{\bar{p}} = (2\pi/L)^{3/2} V_{\bar{p}}$, when $L \to \infty$, where $V_{\bar{p}}$ is a regular function of the continuous variable $\bar{p}$. If this is the case we obtain the continuous limit

$$\hat{H}_{\text{int}} = \int d^3 p V_{\bar{p}} [\hat{\hat{a}}^{\dagger}(\bar{p}) \hat{\hat{b}} + \hat{\hat{b}}^{\dagger} \hat{\hat{a}}(\bar{p})]. \tag{7}$$

For the case of dipolar electromagnetic interaction, the coefficients have the form $g_{\bar{p}} = (2\pi/L)^{3/2} V_{\bar{p}}$, where the $L^{-3/2}$ factor comes from the expansion in normal modes of the quantum field. This is a clear example to understand that we need more than one discrete spectrum becoming continuous to have a well defined model in the continuous limit.

There is an analytic expression which diagonalizes the total Hamiltonian

$$\hat{H} = \hat{H}_{\text{osc}} + \hat{H}_{\text{bath}} + \hat{H}_{\text{int}} = \int d^3 p \omega_{\bar{p}} \hat{\hat{A}}^{\dagger}(\bar{p}) \hat{\hat{A}}(\bar{p}), \tag{8}$$

with the following Bogolyubov transformations [3]

$$\begin{aligned}
\hat{\hat{A}}^{\dagger}(\bar{p}) &= \hat{\hat{a}}^{\dagger}(\bar{p}) + \frac{V_{\bar{p}}}{\eta_{+}(\omega_{\bar{p}})} \left[ \hat{\hat{b}}^{\dagger} + \int d^3 p' \frac{V_{\bar{p}'} \hat{\hat{a}}^{\dagger}(\bar{p}')}{\omega_{\bar{p}} - \omega_{\bar{p}'} + i0} \right], \\
\hat{\hat{b}}^{\dagger} &= \int d^3 p' \frac{V_{\overline{p}'}}{\eta_{-}(\omega_{\bar{p}'})} \hat{\hat{A}}^{\dagger}(\bar{p}'), \\
\hat{\hat{a}}^{\dagger}(\bar{p}) &= \hat{\hat{A}}^{\dagger}(\bar{p}) + \int d^3 p' \frac{V_{\bar{p}'} V_{\bar{p}}}{\eta_{+}(\omega_{\bar{p}'})(\omega_{\bar{p}'} - \omega_{\bar{p}} + i0)} \hat{\hat{A}}^{\dagger}(\bar{p}'), \\
\eta_{\pm}(\omega_{\bar{p}}) &= \omega_{\bar{p}} - \Omega - \int d^3 p' \frac{V^2_{\bar{p}'}}{\omega_{\bar{p}} - \omega_{\bar{p}'} \pm i0}.
\end{aligned} \tag{9}$$

The main goal of quantum theory is to provide a way to compute probabilities for the results of experiments following a given preparation. Therefore the model just presented would not be complete unless we characterize the class of states and observables by suitable definitions. This characterization will be developed in the next two sections.

## 3 Observables

### 3.1 Oscillator Observables

All the observables of the oscillator can be obtained in terms of the corresponding operators $\hat{b}^{\dagger}$ and $\hat{b}$. It is easy to prove from (3) that

$$(\hat{\hat{b}}^{\dagger})^n(\hat{\hat{b}})^m = (\hat{b}^{\dagger})^n(\hat{b})^m \otimes \hat{I}_{\text{bath}}.$$

Therefore, for a regular function $O_{\text{osc}}(x, y)$ we can write

$$O_{\text{osc}}(\hat{\hat{b}}^{\dagger}, \hat{\hat{b}}) = O_{\text{osc}}(\hat{b}^{\dagger}, \hat{b}) \otimes \hat{I}_{\text{bath}},$$

and we have obtained the way in which any operator representing an observable of the oscillator can be lifted to the tensor product space. This class of operators are relevant to study the environment induced decoherence (EID) of the oscillator.

Using the Bogolyubov transformation given in (9) we can write

$$O_{\text{osc}}(\hat{\hat{b}}^{\dagger}, \hat{\hat{b}}) = O_{\text{osc}}(\hat{b}^{\dagger}, \hat{b}) \otimes \hat{I}_{\text{bath}} = O_{\text{osc}}\left(\int d^3p' \frac{V_{\bar{p}'}}{\eta_-(\omega_{\bar{p}'})}\hat{\hat{A}}^{\dagger}(\bar{p}'); \int d^3p' \frac{V_{\bar{p}'}}{\eta_+(\omega_{\bar{p}'})}\hat{\hat{A}}(\bar{p}')\right).$$

If $O_{\text{osc}}(x, y)$ is a regular function, the oscillator observables can be expressed in the following way

$$\hat{O}_{\text{osc}} \otimes \hat{I}_{\text{bath}} = \sum_{n,m} \int d^{3n}k \int d^{3m}p D(\bar{k}_1, \ldots, \bar{k}_n; \bar{p}_1, \ldots, \bar{p}_m) \times \hat{\hat{A}}^{\dagger}(\bar{k}_1) \cdots \hat{\hat{A}}^{\dagger}(\bar{k}_n)\hat{\hat{A}}(\bar{p}_1) \cdots \hat{\hat{A}}(\bar{p}_m), \tag{10}$$

where $D(\bar{k}_1, \ldots, \bar{k}_n; \bar{p}_1, \ldots, \bar{p}_m)$ are regular functions of all the variables $\bar{k}$ and $\bar{p}$.

### 3.2 Bath Observables

The values of global observables of the bath, like energy, momentum or number of particles, increase with the volume of the cubic box and we do not expect well defined values when $L \to \infty$. Therefore this class of observables should not be included in the continuous limit. However, we expect well defined values of *quasi-local observables* of the bath, having the form [17]

$$\hat{B}(\bar{r}) = \int d^3r'_1 \cdots d^3r'_m \int d^3r_1 \cdots d^3r_n \hat{\Psi}^{\dagger}(\bar{r}'_1) \cdots \hat{\Psi}^{\dagger}(\bar{r}'_m) \times b(\bar{r}, \bar{r}'_1 \cdots \bar{r}'_m, \bar{r}_1 \cdots \bar{r}_n)\hat{\Psi}(\bar{r}_1) \cdots \hat{\Psi}(\bar{r}_n),$$

$$\hat{\Psi}^{\dagger}(\bar{r}) = \frac{1}{(2\pi)^{3/2}} \int d^3p \hat{a}^{\dagger}(\bar{p}) \exp(-i\bar{p} \cdot \bar{r}),$$

where $b(\bar{r}, \bar{r}'_1 \cdots \bar{r}'_m, \bar{r}_1 \cdots \bar{r}_n)$ goes to zero if any of the $\bar{r}'_1, \ldots, \bar{r}'_m, \bar{r}_1, \ldots, \bar{r}_n$ does not belong to a neighborhood of $\bar{r}$.

These observables can be represented as

$$\hat{B}(\bar{r}) = \sum_{n,m} \int d^{3m}k \int d^{3n}p B(\bar{r}; \bar{k}_1, \ldots, \bar{k}_m; \bar{p}_1, \ldots, \bar{p}_n) \\ \times \hat{a}^\dagger(\bar{k}_1) \cdots \hat{a}^\dagger(\bar{k}_m) \hat{a}(\bar{p}_1) \cdots \hat{a}(\bar{p}_n), \tag{11}$$

where $B(\bar{r}; \bar{k}_1, \ldots, \bar{k}_m; \bar{p}_1, \ldots, \bar{p}_n)$ are *regular* function. No singular terms seem to appear in the observables of the bath. If we replace the operators $\hat{a}^\dagger$ and $\hat{a}$ by their lifted versions $\hat{\hat{a}}^\dagger$ and $\hat{\hat{a}}$, and then we use (9) to write these operators in terms of $\hat{\hat{A}}^\dagger$ and $\hat{\hat{A}}$, the bath operators can be expressed as

$$\hat{I}_{\text{osc}} \otimes \hat{O}_{\text{bath}} = \sum_{n,m} \int d^{3m}k \int d^{3n}p C(\bar{k}_1, \ldots, \bar{k}_m; \bar{p}_1, \ldots, \bar{p}_n) \\ \times \hat{\hat{A}}^\dagger(\bar{k}_1) \cdots \hat{\hat{A}}^\dagger(\bar{k}_m) \hat{\hat{A}}(\bar{p}_1) \cdots \hat{\hat{A}}(\bar{p}_n), \tag{12}$$

where $C(\bar{k}_1, \ldots, \bar{k}_m; \bar{p}_1, \ldots, \bar{p}_n)$ is a regular function.

### 3.3 General Observables

The expressions (10) and (12) for the oscillator and the bath suggest that a suitable definition for the more general relevant observables of the composed system (oscillator plus bath) is given by

$$\hat{\hat{O}} = \sum_{n,m} \int d^{3n}k \int d^{3m}p O(\bar{k}_1, \ldots, \bar{k}_n; \bar{p}_1, \ldots, \bar{p}_m) \hat{\hat{A}}^\dagger(\bar{k}_1) \cdots \hat{\hat{A}}^\dagger(\bar{k}_n) \hat{\hat{A}}(\bar{p}_1) \cdots \hat{\hat{A}}(\bar{p}_m), \tag{13}$$

where $O(\bar{k}_1, \ldots, \bar{k}_n; \bar{p}_1, \ldots, \bar{p}_m)$ are *regular functions* of the variables $\bar{k}$ and $\bar{p}$.

## 4 States

We only need to define the class of initial states $\hat{\hat{\rho}}^0$ in the Schrödinger representation. We first consider the discrete case and in a second step the case $L \to \infty$. Oscillator and bath are supposed to be initially uncorrelated, i.e.

$$\langle \hat{O}_{\text{osc}} \otimes \hat{O}_{\text{bath}} \rangle = \text{Tr}[\hat{\hat{\rho}}^0 \hat{O}_{\text{osc}} \otimes \hat{O}_{\text{bath}}] = \text{Tr}[\hat{\rho}^0_{\text{osc}} \hat{O}_{\text{osc}}] \, \text{Tr}[\hat{\rho}^0_{\text{bath}} \hat{O}_{\text{bath}}]$$

The oscillator state $\hat{\rho}^0_{\text{osc}}$ presents no difficulty, as it can be represented by a standard density operator. Therefore, for $\hat{O}_{\text{osc}} = (\hat{b}^\dagger)^n (\hat{b})^m$ we can write

$$\text{Tr}[\hat{\rho}^0_{\text{osc}} (\hat{b}^\dagger)^n (\hat{b})^m] \equiv \rho^0_{nm}. \tag{14}$$

The bath state $\hat{\rho}^0_{\text{bath}}$ is more involved. As we wish to analyze the interaction between the oscillator and a thermal bath, let us start considering, for the discrete case, the initial state of the bath at temperature $T$. It is represented by the density operator

$$\hat{\rho}^0_{\text{bath}} = \frac{1}{Z} \exp\left[ -\frac{1}{kT} \hat{H}_{\text{bath}} \right], \tag{15}$$

where $\mathrm{Tr}[\hat{\rho}^0_{\mathrm{bath}}] = 1$, $Z$ is a normalization constant, and $\hat{H}_{\mathrm{bath}} = \sum_{\bar{p}} \omega_{\bar{p}} \hat{a}^\dagger_{\bar{p}} \hat{a}_{\bar{p}}$.

For the thermal state given by (15) the following mean values are obtained [17]

$$\begin{aligned} &\mathrm{Tr}[\hat{\rho}^0_{\mathrm{bath}} \hat{a}^\dagger_{\bar{p}_1} \cdots \hat{a}^\dagger_{\bar{p}_n} \hat{a}_{\bar{p}'_1} \cdots \hat{a}_{\bar{p}'_m}] = \delta_{nm} \sum_{\mathrm{perm}(\bar{p}'_1 \cdots \bar{p}'_n)} \delta_{\bar{p}_1 \bar{p}'_1} \cdots \delta_{\bar{p}_n \bar{p}'_n} f_{\bar{p}_1} \cdots f_{\bar{p}_n}, \\ &f_{\bar{p}} = \frac{1}{\exp(\frac{\omega_{\bar{p}}}{kT}) - 1}, \end{aligned} \tag{16}$$

where the sum is over all possible permutations of the indexes $(\bar{p}'_1, \ldots, \bar{p}'_n)$. All the operators representing observables of the bath can be written in terms of products of creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$.

For a thermal state of the bath the number of exited modes becomes infinite when $L \to \infty$, and we have what is called thermodynamic limit.

The density operator of (15) is not defined when $L \to \infty$, but the mean value given by (16) has a well defined limit

$$\begin{aligned} &\lim_{L\to\infty} \mathrm{Tr}[\hat{\rho}^0_{\mathrm{bath}} \hat{a}^\dagger(\bar{p}_1) \cdots \hat{a}^\dagger(\bar{p}_n) \hat{a}(\bar{p}'_1) \cdots \hat{a}(\bar{p}'_m)] \\ &\quad = \delta_{nm} \sum_{\mathrm{perm}(\bar{p}'_1 \cdots \bar{p}'_n)} \delta^3(\bar{p}_1 - \bar{p}'_1) \cdots \delta^3(\bar{p}_n - \bar{p}'_n) f_{\bar{p}_1} \cdots f_{\bar{p}_n}. \end{aligned} \tag{17}$$

In spite of the fact that the density operator is not defined in the continuous limit, the well defined limit of (17) is useful to *define a state functional* $\rho^0_{\mathrm{bath}}$ such that

$$\begin{aligned} &(\rho^0_{\mathrm{bath}} | \hat{a}^\dagger(\bar{p}_1) \cdots \hat{a}^\dagger(\bar{p}_n) \hat{a}(\bar{p}'_1) \cdots \hat{a}(\bar{p}'_m)) \\ &\quad \equiv \delta_{nm} \sum_{\mathrm{perm}(\bar{p}'_1 \cdots \bar{p}'_n)} \delta^3(\bar{p}_1 - \bar{p}'_1) \cdots \delta^3(\bar{p}_n - \bar{p}'_n) f_{\bar{p}_1} \cdots f_{\bar{p}_n}. \end{aligned} \tag{18}$$

The singular structure of the state functional is a characteristic of the thermodynamic limit. However, a more general singular structure of the initial state is necessary to describe the approach to thermal equilibrium of the bath [17]. We are not dealing with the approach to equilibrium of the bath state in this paper, because this approach cannot be obtained from the model we studied, as there is no interaction between the different bath modes. The thermal equilibrium of the bath we are considering is an initial condition, due to a previous preparation which is not modeled in our approach.

Lifting the previous expressions (14) and (18) to the tensor product Hilbert space, in the limit $L \to \infty$, we obtain an initial state functional $(\rho^0|$ for the composed system oscillator-bath, satisfying

$$\begin{aligned} &(\rho^0 | (\hat{\hat{b}}^\dagger)^r \hat{\hat{a}}^\dagger(\bar{p}_1) \cdots \hat{\hat{a}}^\dagger(\bar{p}_n) (\hat{\hat{b}})^s \hat{\hat{a}}(\bar{p}'_1) \cdots \hat{\hat{a}}(\bar{p}'_m)) \\ &\quad = (\rho^0 | (\hat{b}^\dagger)^r (\hat{b})^s \otimes \hat{a}^\dagger(\bar{p}_1) \cdots \hat{a}^\dagger(\bar{p}_n) \hat{a}(\bar{p}'_1) \cdots \hat{a}(\bar{p}'_m)) \\ &\quad = (\rho^0_{\mathrm{osc}} | (\hat{b}^\dagger)^r (\hat{b})^s)(\rho^0_{\mathrm{bath}} | \hat{a}^\dagger(\bar{p}_1) \cdots \hat{a}^\dagger(\bar{p}_n) \hat{a}(\bar{p}'_1) \cdots \hat{a}(\bar{p}'_m)) \\ &\quad = \rho^0_{rs} \delta_{nm} \sum_{\mathrm{perm}(\bar{p}'_1 \cdots \bar{p}'_n)} \delta^3(\bar{p}_1 - \bar{p}'_1) \cdots \delta^3(\bar{p}_n - \bar{p}'_n) f_{\bar{p}_1} \cdots f_{\bar{p}_n}. \end{aligned} \tag{19}$$

To obtain the last expression we have used the results of (14) and (18).

We can also obtain an expression for the action of the initial state functional on products of the creation and annihilation operators $\hat{\hat{A}}^{\dagger}$ and $\hat{\hat{A}}$,

$$
\begin{aligned}
&(\rho^0|\hat{\hat{A}}^{\dagger}(\bar{p}_1)\cdots\hat{\hat{A}}^{\dagger}(\bar{p}_n)\hat{\hat{A}}(\bar{p}'_1)\cdots\hat{\hat{A}}(\bar{p}'_m)) \\
&\quad = \left(\rho^0 \,\middle|\, \left\{\hat{\hat{a}}^{\dagger}(\bar{p}_1) + \frac{V_{\bar{p}_1}}{\eta_+(\omega_{\bar{p}_1})}\left[\hat{\hat{b}}^{\dagger} + \int d^3p' \frac{V_{\bar{p}'}\hat{\hat{a}}^{\dagger}(\bar{p}')}{\omega_{\bar{p}_1} - \omega_{\bar{p}'} + i0}\right]\right\}\cdots\right),
\end{aligned}
$$

where we have used once again (9). Using the linearity of the state functional and (19), we obtain

$$
\begin{aligned}
&(\rho^0|\hat{\hat{A}}^{\dagger}(\bar{p}_1)\cdots\hat{\hat{A}}^{\dagger}(\bar{p}_n)\hat{\hat{A}}(\bar{p}'_1)\cdots\hat{\hat{A}}(\bar{p}'_m)) \\
&\quad = \delta_{nm} \sum_{\mathrm{perm}(\bar{p}'_1\cdots\bar{p}'_n)} \delta^3(\bar{p}_1 - \bar{p}'_1)\cdots\delta^3(\bar{p}_n - \bar{p}'_n)\rho_0(\bar{p}_1,\ldots,\bar{p}_n) \\
&\qquad + \rho_0(\bar{p}_1\cdots\bar{p}_n;\bar{p}'_1\cdots\bar{p}'_m),
\end{aligned}
\tag{20}
$$

where $\rho_0(\bar{p}_1,\ldots,\bar{p}_n) = f_{\bar{p}_1}\cdots f_{\bar{p}_n}$ and $\rho_0(\bar{p}_1\cdots\bar{p}_n;\bar{p}'_1\cdots\bar{p}'_m)$ are *regular functions*. The just defined observables and states are a generalization of those defined in paper [3] for the particular case $n = m = 2$.

The presence of the Dirac deltas in the previous expression shows that the initial bath states necessarily have a singular structure, and can not be represented by ordinary density operators.

## 5 Time Evolution

### 5.1 The Final State

Taking into account the diagonal form of the Hamiltonian given in (8), we obtain the Heisenberg representation of the creation and annihilation operators

$$\hat{\hat{A}}^{\dagger}(\bar{k},t) = \hat{\hat{A}}^{\dagger}(\bar{k})\exp(it\omega_{\bar{k}}).$$

Them the time dependence of the mean value of an observable of the form given in (13) is given by

$$
\begin{aligned}
(\rho^0|\hat{\hat{O}}\,(t)) = \sum_{n,m}\int d^{3n}k\int d^{3m}p\,O(\bar{k}_1,\ldots,\bar{k}_n;\bar{p}_1,\ldots,\bar{p}_m) \\
\times(\rho^0|\hat{\hat{A}}^{\dagger}(\bar{k}_1)\cdots\hat{\hat{A}}^{\dagger}(\bar{k}_n)\hat{\hat{A}}(\bar{p}_1)\cdots\hat{\hat{A}}(\bar{p}_m)) \\
\times\exp it(\omega_{\bar{k}_1}+\cdots+\omega_{\bar{k}_n}-\omega_{\bar{p}_1}-\cdots-\omega_{\bar{p}_m}).
\end{aligned}
\tag{21}
$$

Therefore using (20)

$$(\rho^0|\hat{\hat{O}}(t)) = \sum_{n,m}\int d^{3n}k\int d^{3m}p\Big[\delta_{nm}\sum_{\text{perm}(\bar{p}'_1\cdots\bar{p}'_n)}\delta^3(\bar{p}_1-\bar{p}'_1)\cdots\delta^3(\bar{p}_n-\bar{p}'_n)f_{\bar{p}_1}\cdots f_{\bar{p}_n}$$
$$+\rho_0(\bar{p}_1\cdots\bar{p}_n;\bar{p}'_1\cdots\bar{p}'_m)\Big]O(\bar{k}_1,\ldots,\bar{k}_n;\bar{p}_1,\ldots,\bar{p}_m)$$
$$\times\exp it(\omega_{\bar{k}_1}+\cdots+\omega_{\bar{k}_n}-\omega_{\bar{p}_1}-\cdots-\omega_{\bar{p}_m}),$$

so using the Riemann_Lebesgue theorem we obtain

$$\lim_{t\to\infty}(\rho^0|\hat{\hat{O}}(t)) = \sum_n\int d^{3n}p f_{\bar{p}_1}\cdots f_{\bar{p}_n}O(\bar{p}_1,\ldots,\bar{p}_n;\bar{p}_1,\ldots,\bar{p}_n)\equiv(\rho^*|\hat{\hat{O}}). \tag{22}$$

The whole system reaches in a weak limit an equilibrium state functional $(\rho^*|$. This would be the best version we can have of SID in this case. The general case will be studied elsewhere.

The time dependent state functional of the oscillator $(\rho_{\text{osc}}(t)|$ (in Schrödinger representation), is obtained by considering observables of the form $\hat{\hat{O}} = \hat{O}_{\text{osc}}\otimes\hat{I}_{\text{bath}}$ and it is defined through the following equation

$$(\rho^0|\hat{\hat{O}}(t)) = (\rho(t)|\hat{\hat{O}}) = (\rho(t)|\hat{O}_{\text{osc}}\otimes\hat{I}_{\text{bath}})\equiv(\rho_{\text{osc}}(t)|\hat{O}_{\text{osc}}). \tag{23}$$

To find an explicit expression for the final state of the oscillator, let us consider

$$\begin{aligned}\hat{\hat{O}} &= (\hat{b}^\dagger)^n(\hat{b})^m\otimes\hat{I}_{\text{bath}} = (\hat{\hat{b}}^\dagger)^n(\hat{\hat{b}})^m\\ &= \int d^3k_1\cdots d^3k_n\int d^3p_1\cdots d^3p_m\\ &\quad\times\frac{V_{\bar{k}_1}\cdots V_{\bar{k}_n}V_{\bar{p}_1}\cdots V_{\bar{p}_n}\hat{\hat{A}}^\dagger(\bar{k}_1)\cdots\hat{\hat{A}}^\dagger(\bar{k}_n)\hat{\hat{A}}(\bar{p}_1)\cdots\hat{\hat{A}}(\bar{p}_m)}{\eta_-(\omega_{\bar{k}_1})\cdots\eta_-(\omega_{\bar{k}_n})\eta_+(\omega_{\bar{p}_1})\cdots\eta_+(\omega_{\bar{p}_m})}.\end{aligned} \tag{24}$$

Therefore using (22) and (24) we obtain

$$\lim_{t\to\infty}(\rho(t)|(\hat{\hat{b}}^\dagger)^n(\hat{\hat{b}})^m) = \delta_{nm}\left[\int d^3k\frac{V_{\bar{k}}^2 f_{\bar{k}}}{\eta_-(\omega_{\bar{k}})\eta_+(\omega_{\bar{k}})}\right]^n.$$

Then from (23)

$$\lim_{t\to\infty}(\rho_{\text{osc}}(t)|(\hat{b}^\dagger)^n(\hat{b})^m) = (\rho^*_{\text{osc}}|(\hat{b}^\dagger)^n(\hat{b})^m).$$

This last equation shows that the oscillator reaches, in a weak limit, a final state of equilibrium. For a small interaction we have [3]

$$\frac{V_{\bar{k}}^2}{\eta_-(\omega_{\bar{k}})\eta_+(\omega_{\bar{k}})}\cong\frac{\delta(\omega_{\bar{k}}-\Omega)}{4\pi\Omega^2}.$$

Considering also the expression for $f_{\bar{k}}$ given in (16) we finally obtain

$$(\rho^*_{\text{osc}}|(\hat{b}^\dagger)^n(\hat{b})^m) = \delta_{nm}\left[\frac{1}{\exp(\frac{\Omega}{kT})-1}\right]^n.$$

These results correspond to an oscillator at temperature $T$, for which the density operator is $\hat{\rho}^*_{\text{osc}} \propto \exp[-\frac{1}{kT}\Omega\hat{b}^\dagger\hat{b}]$. Obviously the final state can be represented by a density operator which is diagonal in the energy basis of the oscillator.

### 5.2 Decoherence Time

We can show through this model that the characteristic time for approaching equilibrium strongly depends on the observable.

Let us consider the oscillator observable $\hat{\hat{O}} = (\hat{b}^\dagger)^n(\hat{b})^m \otimes \hat{I}_{\text{bath}}$. Taking into account equations (24) and the general form of the state functional given by (20) we obtain

$$\begin{aligned}(\rho(t)|\hat{\hat{O}}) = \delta_{nm}\left[\int d^3k \frac{V_{\bar{k}}^2 f_{\bar{k}}}{\eta_-(\omega_{\bar{k}})\eta_+(\omega_{\bar{k}})}\right]^n \\ + \int d^3k_1 \cdots d^3k_n \int d^3p_1 \cdots d^3p_m \\ \times \frac{V_{\bar{k}_1}\cdots V_{\bar{k}_n} V_{\bar{p}_1}\cdots V_{\bar{p}_n} B(\bar{p}_1\cdots\bar{p}_n;\bar{p}'_1\cdots\bar{p}'_m)}{\eta_-(\omega_{\bar{k}_1})\cdots\eta_-(\omega_{\bar{k}_n})\eta_+(\omega_{\bar{p}_1})\cdots\eta_+(\omega_{\bar{p}_m})} \\ \times \exp it(\omega_{\bar{k}_1}+\cdots+\omega_{\bar{k}_n}-\omega_{\bar{p}_1}-\cdots-\omega_{\bar{p}_m}),\end{aligned}$$

where $B(\bar{p}_1\cdots\bar{p}_n;\bar{p}'_1\cdots\bar{p}'_m)$ are regular functions depending on the temperature of the bath.

If the analytic extension $\eta_+(z)$ from the upper to the lower complex half plane of $\eta_+(\omega)$ has a simple zero at $z = z_0$, where $z_0$ is close to the real axis, the time dependence of the last expression will be dominated by a factor of the form $\exp it(nz_0^* - mz_0)$. Therefore the time dependent part of the mean value decays almost exponentially.

Let us also consider an observable of the bath, for example $\hat{\hat{O}} = \hat{I}_{\text{osc}} \otimes \hat{a}^\dagger(\bar{p})\hat{a}(\bar{k}) = \hat{\hat{a}}^\dagger(\bar{p})\hat{\hat{a}}(\bar{k})$. It is possible to show that the time dependent part of the mean value of this observable will approach to zero, but this approach is not dominated by an exponential decay.

All these features are reminiscent of what happened in the model of paper [12].

## 6 Conclusions

If irreversibility is *defined* as the existence of a well defined state when $t \to \infty$, there is no irreversibility for systems with discrete energy spectrum. SID cannot be implemented in a model with discrete spectrum of energy, because Riemann–Lebesgue theorem can not be used if there are no integrals.

If the gap between the discrete eigenvalues of the Hamiltonian becomes very small for increasing values of the box size $L$, it may be possible to find an *approximate* irreversibility, where there is a time evolution to *something like* an equilibrium state, but not for too large times. A system with discrete spectrum is quasi periodic, and therefore a recurrence time exists for which the state of the system is arbitrary close to the initial state. However, the recurrence time may be very large for a big value of the size $L$ of the box.

If we have a Hamiltonian, *and* a class of states, *and* a class of observables for which there are well defined limit expressions when $L \to \infty$, and only in this case, a "limit quantum model" with continuous spectrum could be obtained. In this case SID can be implemented and it can be expected that the predictions for the time evolution may *approximately* coincide with the predictions for the discrete model, for a big $L$ and a not too big $t$, as it is shown for the Friedrichs model by Gaioli et al. [16].

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